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Propagator on the h-deformed Lobachevsky plane

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Abstract. The action of the isometry algebra $U_h(sl(2))$ on the h-deformed Lobachevsky plane is found. The invariant distance and the invariant two-point functions are shown to agree precisely with the classical ones. The propagator of the Laplacian is calculated explicitly. It is invariant only after adding a 'non-classical' sector to the Hilbert space.

1. Introduction

The h-deformed Lobachevsky plane was introduced by Demidov *et al* [6] and by Manin [11]. Its function algebra is covariant under $Sl_h(2, \mathbb{R})$, which is a triangular Hopf algebra, sometimes called the Jordanian deformation of $Sl(2, \mathbb{R})$. As opposed to the q-deformed quantum groups, it is triangular, which means that the deformation from the classical case is less severe. In fact, it is known that $U_h(sl(2))$ is related to its undeformed counterpart by a twist [1, 10]. While this might suggest that the deformation is almost trivial in some sense, the problem of defining suitable spaces of functions, in particular Hilbert spaces, which can be relevant to physical systems is not at all trivial. In fact, it will turn out that in order to find an invariant propagator, a certain 'non-classical' sector must be added to the Hilbert space, which disappears in the classical limit.

The main goal of this paper is to calculate explicitly the invariant propagator on the h-deformed Lobachewski plane. The first observation is that the well known covariance algebra $U_h(sl(2))$ does not preserve the metric structure, but only the symplectic structure. Therefore, in section 2, we first determine the three-parameter 'group' of isometries, which turns out to be again $U_h(sl(2))$, but with a different action on the space which corresponds to the well known fractional transformations of the upper half-plane.

In order to define n-point 'functions' in a covariant way, in section 3 we introduce braided copies of the h-deformed Lobachevsky plane. This allows one to determine invariant functions, and in particular the invariant distance between two 'points'. The distance turns out to involve only a commutative subalgebra of the complete algebra, and agrees precisely with the classical one.

In section 5, we calculate the propagator of the h-deformed Laplacian explicitly. When based on a naive generalization of the Hilbert space of modes of the undeformed case, it turns out that the propagator is not invariant under $U_h(sl(2))$. It does become invariant only after adding another, 'non-classical' sector to the Hilbert space. This situation is reminiscent of a similar phenomenon on q-deformed quantum spaces [8] and shows that the h-deformation is not quite a trivial one. The propagator on the extended Hilbert space then turns out to agree formally with the classical one.

This result should be compared with that of a recent work [12] where the propagator on the the h-deformed Lobachevsky plane has been found to be finite provided one uses the *undeformed* tensor product. Of course, this breaks the covariance under $U_h(sl(2))$. In our covariant treatment, the propagator turns out not to be regularized. This means that either the h-deformation is not strong enough to regularize the UV divergences, or that the different copies of the Hilbert space should *not* be implemented via the braided tensor product.

2. The isometries of the h-deformed Lobachevsky plane

The h-deformed Lobachevsky plane [2] can be defined [5] to be the formal $*$-algebra $\mathcal{A}$ generated by two Hermitian elements x and y which satisfy the commutation relation

$$[x, y] = -2\mathrm{i}hy \tag{2.1}$$

where $h \in \mathbb{R}$ and the factor -2 is present for historical reasons. We shall suppose that $h > 0$. Throughout this paper, a 'function' on the h-deformed Lobachevsky plane is understood to be an element of $\mathcal{A}$ or a suitable completion thereof.

Using the variable $z = x + \mathrm{i}y$, this becomes

$$[z, \bar{z}] = 2\mathrm{i}h(z - \bar{z}). \tag{2.2}$$

Introducing variables r, s by $x = rs^{-1} + \frac{1}{2}\mathrm{i}h$ and $y = s^{-2}$, the above commutation relation becomes

$$[r, s] = \mathrm{i}hs^2. \tag{2.3}$$

In terms of these variables, it is easy to check that the algebra is covariant under $Sl_h(2, \mathbb{R})$, i.e. there is a coaction $\Delta : \mathcal{A} \to \mathrm{Fun}(Sl_h(2, \mathbb{R})) \otimes \mathcal{A}$ given by

$$\begin{pmatrix} r \\ s \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \dot{\otimes} \begin{pmatrix} r \\ s \end{pmatrix} \tag{2.4}$$

where the algebra $\mathrm{Fun}(Sl_h(2, \mathbb{R}))$ is the h-deformed (Hopf) $*$-algebra of functions on $Sl(2, \mathbb{R})$ generated by the Hermitian elements $A, B, C, D,$ with relations

$$\begin{aligned}
[A, B] &= \mathrm{i}h\delta - \mathrm{i}hA^2 \\
[A, C] &= \mathrm{i}hC^2 \\
[A, D] &= \mathrm{i}hCD - \mathrm{i}hCA \\
[B, C] &= \mathrm{i}hCD + \mathrm{i}hAC \\
[B, D] &= \mathrm{i}hD^2 - \mathrm{i}h\delta \\
[C, D] &= -\mathrm{i}hC^2
\end{aligned}$$

where the quantum determinant

$$\delta = AD - CB - \mathrm{i}hCD = DA - CB - \mathrm{i}hCA \tag{2.5}$$

is central and set equal to one.

The R-matrix associated with this quantum group, which solves the quantum Yang–Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{2.6}$$

is given by

$$\hat{R} = \begin{pmatrix} 1 & -\mathrm{i}h & \mathrm{i}h & -h^2 \\ 0 & 0 & 1 & \mathrm{i}h \\ 0 & 1 & 0 & -\mathrm{i}h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

It is triangular, i.e. $\hat{R}^2 = 1$, which also holds for the higher representations. The associated calculus and Laplacian have been worked out elsewhere [2, 4, 5]. We will thus be brief here.

The covariant differential calculus $(\Omega^*(\mathcal{A}_h), d)$ on a quantum plane can be found [2] by the method of Wess and Zumino [18]. For $r^i = (r, s)$ and $\xi^i = \mathrm{d}r^i = (\xi, \eta)$ we have

$$\begin{aligned} &r^a r^b = \hat{R}^{ab}{}_{cd} r^c r^d && r^a \xi^b = \hat{R}^{ab}{}_{cd} \xi^c r^d \\ &\xi^a \xi^b = -\hat{R}^{ab}{}_{cd} \xi^c \xi^d && \partial_a x^b = \delta_a^b + \hat{R}^{bd}{}_{ac} x^c \partial_d. \end{aligned} \tag{2.8}$$

Explicitly, the second and third equations are

$$\begin{aligned} &[r, \xi] = -\mathrm{i}h\xi s + \mathrm{i}h\eta r - h^2 \eta s && [r, \eta] = \mathrm{i}h\eta s \\ &[s, \xi] = -\mathrm{i}h\eta s && [s, \eta] = 0 \end{aligned} \tag{2.9}$$

and

$$\xi^2 = \mathrm{i}h\xi\eta \qquad \xi\eta = -\eta\xi \qquad \eta^2 = 0. \tag{2.10}$$

One can also introduce [5] a frame or Stehbein θ^a defined by

$$\theta^1 = y^{-1}\,\mathrm{d}x \qquad \theta^2 = y^{-1}\,\mathrm{d}y. \tag{2.11}$$

They satisfy the commutation relations

$$f\theta^a = \theta^a f \qquad f \in \mathcal{A}_h \tag{2.12}$$

as well as the quadratic relations

$$(\theta^1)^2 = 0 \qquad (\theta^2)^2 = 0 \qquad \theta^1\theta^2 + \theta^2\theta^1 = 0. \tag{2.13}$$

More details of this have been given elsewhere [4, 5]. One can also define a Hodge star operator as

$$*(\theta^1) = \theta^2 \qquad *(1) = \theta^1\theta^2 \tag{2.14}$$

we will see in section 3 that this is also the correct covariant definition.

By construction, the coaction of $\mathrm{Fun}_h(Sl(2, \mathbb{R}))$ on $\mathcal{A}$ preserves the symplectic structure; however, it is not the group of isometries. This is so even classically, as was already noted in

[5]. To find the correct isometries, consider first the commutative limit, where the metric is $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. The isometries are the well known fractional transformations

$$z \to z' = \frac{Az + B}{Cz + D} \qquad \text{with} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Fun}(Sl(2, \mathbb{R}))$$

where $z = x + \mathrm{i}y$. In this section, we will find a similar transformation in the non-commutative case, such that the isometries have the structure of $SL_h(2, \mathbb{R})$, but with a different (co)action than the symplectic one (2.4). The metric will turn out to be the same as in the commutative case.

Since we later wish to determine the functions which are invariant under the isometries, it is more useful to have an action of the universal enveloping algebra $U_h(sl(2))$ on $\mathcal{A}$ rather than a coaction of $\mathrm{Fun}_h(Sl(2))$. Since these are dually paired Hopf algebras, a left (respectively, right) coaction of $\mathrm{Fun}_h(Sl(2))$ corresponds to a right (respectively, left) action of $U_h(sl(2))$; see, for example, [14]. The resulting cross-product algebra is given in (2.24); we take a small detour and explain the steps leading to this algebra.

To find this dual action, we look for variables such that the above fractional transformation of z becomes linear. First, we introduce different variables z_1, z_2 for $\mathcal{A}$ which satisfy

$$[z_1, z_2] = 2\mathrm{i}h(z_1 - z_2) \tag{2.15}$$

with star-structure $z_1^* = \overline{z_1} = z_2 + \mathrm{i}h$; we use a bar to denote the star of z. Then (2.2) is recovered for $z = z_1 + \frac{1}{2}\mathrm{i}h$. One can easily check (and it will become evident below) that this is consistent with the following coaction of $\mathrm{Fun}_h(Sl(2))$:

$$z_i \to (Az_i + B)(Cz_i + D)^{-1} \tag{2.16}$$

for $i = 1, 2$. A similar coaction for certain q-deformed algebras has been considered in [16]. While this form is very appealing, it is somewhat formal, and it is not immediately clear how to translate it into an action of $U_h(sl(2))$ which will be needed below. To find this, we introduce yet another set of (auxiliary) generators. Consider u_i, v_i with $[u_i, v_i] = \mathrm{i}hv_i^2$ for $i = 1, 2$, which are covariant under the linear coaction of $\mathrm{Fun}_h(Sl(2))$ as in (2.3) and (2.4), and let $z_i = u_i v_i^{-1}$. Furthermore, we impose the commutation relations

$$\begin{aligned} [u_1, u_2] &= \mathrm{i}hu_1v_2 - \mathrm{i}hv_1u_2 + h^2v_1v_2 \\ [u_1, v_2] &= \mathrm{i}hv_1v_2 \\ [v_1, u_2] &= -\mathrm{i}hv_1v_2 \\ [v_2, v_2] &= 0. \end{aligned} \tag{2.17}$$

They are consistent with the linear coaction of $\mathrm{Fun}_h(Sl(2))$ as will be explained in the next section, and imply (2.15). The star structure $u_1^* = u_2$, $v_1^* = v_2$ implies $z_1^* = \overline{z_1} = z_2 + \mathrm{i}h$ as above, and the linear coaction on u_i, v_i obviously induces (2.16). In this linear form, we can find the dual action of $U_h(sl(2))$, and then restrict it to the original algebra generated by $z, \overline{z}$.

We recall the definition of $U_h(sl(2))$: it is the Hopf algebra with generators $\{J^{\pm}, J^3\}$ and relations [15]

$$\begin{aligned} [J^3, J^+] &= 2h^{-1}\sin(hJ^+) \\ [J^3, J^-] &= -[\cos(hJ^+)J^- + J^-\cos(hJ^+)] \\ [J^+, J^-] &= J^3 \end{aligned} \tag{2.18}$$

and

$$\Delta J^+ = J^+ \otimes 1 + 1 \otimes J^+ \qquad \Delta J^j = J^j \otimes \mathrm{e}^{-\mathrm{i}hJ^+} + \mathrm{e}^{\mathrm{i}hJ^+} \otimes J^j$$
$$\varepsilon(X) = 0 \qquad S(X) = -\mathrm{e}^{-\mathrm{i}hJ^+} X \mathrm{e}^{\mathrm{i}hJ^+} \tag{2.19}$$

where $j \in \{-, 3\}$ and $X \in \{J^+, J^-, J^3\}$. This is obtained from the result of Ohn [15] by the replacement $h \to -\mathrm{i}h$. It is a $*$-Hopf algebra with the reality structure

$$(J^\pm)^* = -J^\pm \qquad (J^3)^* = -J^3 \tag{2.20}$$

which defines $U_h(sl(2, \mathbb{R}))$. Introducing

$$G = \mathrm{e}^{-\mathrm{i}hJ^+} \tag{2.21}$$

this becomes

$$\begin{aligned} &[G, J^3] = 1 - G^2 \\ &[J^3, J^-] = -\tfrac{1}{2}\big[(G + G^{-1})J^- + J^-(G + G^{-1})\big] \\ &[G, J^-] = -\tfrac{1}{2}\mathrm{i}h(GJ^3 + J^3G) \end{aligned} \tag{2.22}$$

and

$$\Delta G = G \otimes G \qquad \Delta J^j = J^j \otimes G + G^{-1} \otimes J^j$$
$$\varepsilon(X) = 0 \qquad S(X) = -GXG^{-1} \tag{2.23}$$

where $j \in \{-, 3\}$ and $X \in \{J^+, J^-, J^3\}$. Given a (left or right) action of $U_h(sl(2))$ on $\mathcal{A}$, one can always define a cross-product algebra $U_h(sl(2)) \ltimes \mathcal{A}$. As a vector space, this is $U_h(sl(2)) \otimes \mathcal{A}$, equipped with an algebra structure defined by $ua = (u_{(1)} \cdot a)u_{(2)}$, where the dot denotes the left action of $u \in U_h(sl(2))$ on a representation; similarly for a right action. Here $u_{(1)} \otimes u_{(2)}$ denotes the coproduct of u. Conversely, the left action $u \cdot a$ can be extracted by commuting u to the right and then applying the counit ε of $U_h(sl(2))$ from the right.

The dual pairing of $U_h(sl(2))$ with $\mathrm{Fun}_h(Sl(2))$ has been given in [3]. Using this, it is easy to find the dual action of $U_h(sl(2))$ on the variables u_1, v_1, u_2, v_2. This defines a cross-product algebra as explained above, which can be expressed in terms of the original variable z. The resulting algebra is

$$\begin{aligned} &[z, J^+] = -1 \\ &[z, J^-G] = \left(z^2 - \mathrm{i}hz + \tfrac{1}{4}h^2\right) G^2 \\ &[z, J^3G] = (2z - \mathrm{i}h)\, G^2 \end{aligned} \tag{2.24}$$

and the same relations for $\bar{z}$. One can check explicitly that this algebra is consistent with the relations (2.2), the relations of $U_h(sl(2))$, and also with the star structure $z^* = \bar{z}$. Therefore, it is a consistent cross-product algebra for the h-deformed Lobachevsky plane. In terms of x and y with $z = x + \mathrm{i}y, \bar{z} = x - \mathrm{i}y$, it becomes

$$\begin{aligned} &[x, J^+] = -1 \qquad [y, J^+] = 0 \\ &[x, J^-G] = \left(x^2 - y^2 - \mathrm{i}hx + \tfrac{1}{4}h^2\right) G^2 \\ &[y, J^-G] = (2xy + \mathrm{i}hy)\, G^2 \\ &[x, J^3G] = (2x - \mathrm{i}h)\, G^2 \\ &[y, J^3G] = 2yG^2. \end{aligned} \tag{2.25}$$

With

$$D_x f(x) := \frac{f(x) - f(x-2\mathrm{i}h)}{2\mathrm{i}h} \tag{2.26}$$

one finds for functions (power series) in x and y

$$\begin{aligned}
&[f(x), J^+] = -\partial_x f(x) \qquad [f(y), J^+] = 0 \\
&[f(x), J^3 G] = (2x - \mathrm{i}h) D_x f(x)\, G^2 \qquad [f(y), J^3 G] = 2y\partial_y f(y)\, G^2 \\
&[f(x), J^- G] = \left(\left(x^2 - \mathrm{i}hx + \tfrac{1}{4}h^2\right) D_x f(x) - D_x f(x+2\mathrm{i}h) y^2\right) G^2 \\
&[f(y), J^- G] = (2xy\partial_y f(y) + \mathrm{i}h(2(y\partial_y)^2 - y\partial_y) f(y))\, G^2
\end{aligned} \tag{2.27}$$

and

$$G f(x) = f(x - \mathrm{i}h)\, G.$$

We claim that this cross-product algebra implements the isometry algebra $U_h(sl_h(2))$ on the h-deformed Lobachevsky plane. In the limit $h \to 0$, the generators J^+, $-J^3 G$, and $-J^- G$ clearly become the classical generators ∂_x, $2x\partial_x + 2y\partial_y$, and $(x^2 - y^2)\partial_x + 2xy\partial_y$ of the algebra $sl(2, \mathbb{R})$ of isometries. Moreover, in section 3.1, we shall find an explicit expression for the h-deformed distance, which is invariant under the action (2.25) of $U_h(sl_h(2))$. The necessary tools will be provided in the next section.

3. Braided copies of the h-deformed Lobachevsky plane

In order to write down functions of several variables such as n-point functions, one should introduce several copies of the algebra of functions, and combine them into a bigger algebra. Recall the classical case: if $\mathcal{A}$ is a representation of some Lie algebra $\mathfrak{g}$ compatible with the algebra structure of $\mathcal{A}$, more precisely $\mathcal{A}$ is a $\mathfrak{g}$-module algebra, this is easy to do: define $\mathcal{A}^{\otimes} := \mathcal{A} \otimes \cdots \otimes \mathcal{A}$, and let $\mathcal{A}^{(n)} := 1 \otimes \cdots \otimes \mathcal{A} \otimes 1 \otimes \cdots$ be the nth copy of $\mathcal{A}$. Then $\mathcal{A}^{\otimes}$ is naturally an algebra (the tensor product algebra) by componentwise multiplication, and $\mathcal{A}^{(n)}$ commutes with $\mathcal{A}^{(m)}$ if $n \neq m$. $\mathcal{A}^{\otimes}$ carries the tensor product representation of $\mathfrak{g}$, and its algebra structure is compatible with this representation.

If $\mathcal{A}$ is covariant under a Hopf algebra $\mathcal{U}$ which is not co-commutative, this standard algebra structure on $\mathcal{A}^{\otimes}$ is not compatible with the coaction. However, if the Hopf algebra is quasitriangular with universal R-'matrix' $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{U} \otimes \mathcal{U}$ (in shorthand notation), then there is a standard way to define a modified algebra structure on $\mathcal{A}^{\otimes}$, the so-called 'braided tensor product' [13]: it is defined by $a^{(n)} a^{(m)} := 1 \otimes \cdots \otimes a^{(n)} \otimes 1 \otimes \cdots \otimes a^{(m)} \otimes 1 \ldots$ if $n \leqslant m$, and $a^{(n)} a^{(m)} := 1 \otimes \cdots \otimes \mathcal{R}_1 \cdot a^{(m)} \otimes 1 \otimes \cdots \otimes \mathcal{R}_2 \cdot a^{(n)} \otimes 1 \ldots$ if $n > m$, where $a^{(n)} \in \mathcal{A}^{(n)}$. This is compatible with the action of the quasitriangular Hopf algebra $\mathcal{U}$; to avoid confusion, we will denote it by $\mathcal{A}^{\otimes_h}$.

Since $U_h(sl(2))$ is in fact triangular, i.e. $\mathcal{R}_{12}\mathcal{R}_{21} = \mathbf{1}$, the above definition can be written as a commutation relation $a^{(n)} a^{(m)} = (\mathcal{R}_1 \cdot a^{(m)})(\mathcal{R}_2 \cdot a^{(n)})$ in $\mathcal{A}^{\otimes_h}$ whenever $n \neq m$. This is a considerable simplification over the quasitriangular case, where one has to distinguish between $n > m$ and $n < m$. Notice that the commutation relations between functions and the generators of forms in the calculus (2.8) are precisely of this kind.

For the h-deformed Lobachevsky plane, we have two different actions of $U_h(sl(2))$ acting on it, the symplectic one dual to (2.4) and the (tentative) isometries corresponding to (2.24) found in the previous section. Thus it is not clear *a priori* how to proceed. What we will do is to define first the braided algebra $\mathcal{A}^{\otimes_h}$ for the symplectic action since that one is much simpler, and verify that it in fact compatible with the action of the isometries as well; this is not obvious *a priori*.

For simplicity, introduce just two copies of $\mathcal{A}$, i.e. $x = x \otimes 1$ and $x' = 1 \otimes x$. In terms of the variables r, s of section 2 with $x = rs^{-1} + \frac{1}{2}\mathrm{i}h$ and $y = s^{-2}$, this definition leads to

$$
\begin{aligned}
[r, r'] &= \mathrm{i}hrs' - \mathrm{i}hsr' + h^2 s's \\
[r, s'] &= \mathrm{i}hs's \\
[s, r'] &= -\mathrm{i}hs's \\
[s, s'] &= 0
\end{aligned}
$$

which is the same as (2.17). In terms of x and y, this becomes

$$
\begin{aligned}
[x, x'] &= 2\mathrm{i}hx - 2\mathrm{i}hx' \\
[x, y'] &= -2\mathrm{i}hy' \\
[y, x'] &= 2\mathrm{i}hy \\
[y, y'] &= 0.
\end{aligned}
\tag{3.1}
$$

It is somewhat disturbing that the commutator of x and x' does not vanish as $x - x'$ becomes large, but this is required by covariance; we will come back to that later. In the complex variables $z = x + \mathrm{i}y$, $z' = x' + \mathrm{i}y'$, one obtains

$$
[z, z'] = 2\mathrm{i}h(z - z') \tag{3.2}
$$

which explains the relation (2.15). This algebra is by construction consistent with the coaction of the 'symplectic' $\mathrm{Fun}(Sl_h(2))$ equation (2.4), respectively, its dual. It can now be checked by a lengthy but straightforward calculation that these relations are also compatible with (2.24), extended to both copies z and z'. This is not obvious *a priori*. A somewhat similar observation has been made [16] for the q-deformed case in terms of the fractional transformations considered in section 2.

The concept of braided copies of a covariant algebra is also relevant if one tries to define a Fock space of creation and annihilation operators which are covariant under some quantum group. In general, it is not obvious then how to define a totally symmetric or antisymmetric Hilbert space, since the deformed analogue of the permutation operator, $\hat{R}$, has eigenvalues which are different from ± 1. In the triangular case, this problem does not occur, since $\hat{R}^2 = \mathbf{1}$ by definition. From this point of view, the triangular case seems particularly well suited to formulate a quantum field theory. Perhaps, however, triangular Hopf algebras are not a 'sufficiently' non-trivial deformation in order to improve the UV behaviour of the commutative limit. To obtain some insight into this question was one of the motivations of the present work.

3.1. Invariant distance

To make the algebra (3.1) more transparent, we define

$$
\delta x = x - x' \qquad \delta y = y - y'
$$

and

$$
\overline{x} = \tfrac{1}{2}(x + x') \qquad \overline{y} = \tfrac{1}{2}(y + y').
$$

In terms of these variables, one finds as only non-zero commutators

$$\begin{aligned}[\overline{x}, \overline{y}] &= -2\mathrm{i}h\overline{y} \\ [\overline{x}, \delta x] &= -2\mathrm{i}h\delta x \\ [\overline{x}, \delta y] &= -2\mathrm{i}h\delta y.\end{aligned} \tag{3.3}$$

Notice that these are the same relations as for the calculus where δx, δy are replaced by $\mathrm{d}x$ and $\mathrm{d}y$. In particular, $\overline{y}^{-1}\delta x$ and $\overline{y}^{-1}\delta y$ play the role of the Stehbein (2.11), and they commute with $\overline{x}$ and $\overline{y}$. Thus there is only one non-trivial commutator among the four generators of $\mathcal{A} \otimes_h \mathcal{A}$ as opposed to the case $\mathcal{A} \otimes \mathcal{A}$, where the propagator is regularized [12]. In particular, it is somewhat counterintuitive that the 'relative' and 'average' coordinates do *not* mutually commute (cf [12]); again, this is forced upon us by the covariance requirement.

The geodesic distance of two points (x, y) and (x', y') on the classical Lobachavsky plane is given by [9]

$$d = \cosh^{-1}\left(1 + \frac{1}{2yy'}((\delta x)^2 + (\delta y)^2)\right). \tag{3.4}$$

The subalgebra generated by $\overline{y}$, δx, δy is Abelian in the h-deformed case as well, and it can be checked that the same expression is also invariant under the h-deformed isometries. In fact,

$$y^{-1}y'^{-1}((\delta x)^2 + (\delta y)^2)$$

commutes with $U_h(sl(2))$ in the cross-product algebra (2.24), and together with 1 generates the centre of $\mathcal{A}\otimes_h\mathcal{A}$. Again, this is more easily seen in terms of the fractional transformation of section 2, but less rigorous. Thus we shall define (3.4), which is an invariant two-point function, to be the invariant distance function on the h-deformed Lobachevsky plane.

Of course the same considerations apply for the case of several variables. It is then clear that the set of invariant n-point functions is the same as classically, i.e. they are precisely the functions which depend only on the relative distances of any pairs of variables, defined as in (3.4).

One can check that the Hodge star (2.14) is invariant under the isometries as well.

4. Invariant functionals and inner products

In quantum mechanics, symmetries are implemented as unitary transformations on a Hilbert space. To realize this and similarly to define invariant propagators, one has to find a positive definite inner product on a vector space which is invariant under the symmetry. Such invariant inner products are naturally obtained from a positive state, which should satisfy

$$\langle u \cdot a\rangle = \varepsilon(u)\langle a\rangle \tag{4.1}$$

and $\langle a\rangle^* = \langle a^*\rangle$, where a is an element of a star algebra $\mathcal{A}$ and u is an element of a symmetry (Hopf) algebra $\mathcal{U}$. Since we are considering spaces of functions on a (non-commutative) manifold, this can be considered as an invariant functional. Of course, $\langle\ \rangle$ is defined only on a certain subset of 'measurable' elements of a suitable completion of $\mathcal{A}$, as classically. This will become clear in our example.

It is useful to formulate this within the framework of cross-product algebras. Any state (functional) on $\mathcal{A}$ defines a state on $U_h(sl(2)) \ltimes \mathcal{A}$, written as $\langle ua\rangle$, by $\langle ua\rangle := \epsilon(u)\langle a\rangle$. If the state on $\mathcal{A}$ is invariant, this implies using some standard identities of Hopf algebras that $\langle uav\rangle = \epsilon(u)\epsilon(v)\langle a\rangle$ for any $u, v \in U_h(sl(2))$, and in particular

$$\langle[u, a]\rangle = 0 \tag{4.2}$$

for any $u \in U_h(sl(2))$. Conversely, the state $\langle\ \rangle$ on $\mathcal{A}$ is invariant if (4.2) holds. The latter form is quite intuitive, and is well suited for our situation. We will work with this formalism from now on.

As usual, each invariant state induces an invariant inner product as follows:

$$\langle f, g\rangle = \langle f^* g\rangle. \tag{4.3}$$

It is invariant, because $\langle f, u \cdot g\rangle = \langle f^* u g\rangle = \langle (u^* \cdot f)^* g\rangle$, using $u \cdot f = u_1 f S u_2$ and standard identities of Hopf-$*$ algebras.

The conditions (4.2) for the subalgebra of $U_h(sl(2))$ generated by G, J^-, J^3 are

$$\langle [f, G]\rangle = 0 \tag{4.4}$$

$$\langle [f, J^3 G]\rangle = 0 \tag{4.5}$$

$$\langle [f, J^- G]\rangle = 0 \tag{4.6}$$

for $f \in \mathcal{A}$. We will write the elements f in the form $f(x|y) = \sum f_{n,m} x^n y^m$, i.e. with x to the left of y. After some calculations using (2.27), they reduce to the following two requirements:

$$\langle f(x|y)\rangle = \langle f(x + \mathrm{i}h|y)\rangle \tag{4.7}$$

and

$$\left\langle y \frac{\partial}{\partial y} f(x|y)\right\rangle = \langle f(x|y)\rangle. \tag{4.8}$$

Here $\frac{\partial}{\partial y} f(x|y)$ is the ordinary differentiation with respect to y, after ordering the variables as above. It turns out that (4.6) is a consequence of (4.4) and (4.5). This is similar to the classical case, where the invariant integral is also uniquely determined by two isometries, and automatically respects the third. The only difference to the classical case is that the translation invariance with respect to x is imposed only for a finite displacement rather that for all. This of course comes from restricting ourselves to the algebra generated by G, J^3, J^- rather than J^+, J^3, J^-, which is consistent in the h-deformed case only.

One invariant functional satisfying these conditions is now obvious: it is simply the classical one. That is, consider the space $L^1(\mathbb{R}^2_+, \mathrm{d}\mu)$ of functions $f(x, y)$ on the upper half-plane which are integrable with respect to the measure $\mathrm{d}\mu = y^{-2}\,\mathrm{d}x\,\mathrm{d}y = \theta^1\theta^2$. We write the functions (or more precisely a dense set of analytic functions in $L^1(\mathbb{R}^2_+, \mathrm{d}\mu)$) in the form $f(x|y)$ so that they define elements in (a completion of) $\mathcal{A}$, and we set

$$\langle f(x|y)\rangle_{(0)} := \int f(x, y)\,\mathrm{d}\mu. \tag{4.9}$$

Invariance under G follows by analytic continuation in x, e.g. using the basis of Hermite functions in x. In this way, we obtain a space of functions on the h-deformed Lobachevsky plane which is isomorphic to $L^1(\mathbb{R}^2, \mathrm{d}\mu)$. The corresponding Hilbert space will be defined explicitly in the next section. At this point, the h-deformed case indeed appears to be isomorphic to the undeformed case.

It will turn out, however, that this 'classical' Hilbert space is not sufficient to obtain invariant propagators, but we shall be able to introduce 'extended' Hilbert spaces by taking advantage of the weaker requirement (4.7).

Finally, one can define an integral of 2-forms $\alpha = f(x|y)\theta^1\theta^2$ by

$$\int \alpha = \langle f(x|y)\rangle.$$

It is easy to see that invariance of $\langle\ \rangle$ under G, J^- and J^3 is equivalent to Stokes' theorem, $\int \mathrm{d}\omega = 0$, and that the adjoint of d in the usual sense is indeed $\delta = *\mathrm{d}*$.

5. The propagator

The h-deformed Laplacian can be defined [4] as $-\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. In this form, the invariance under the isometries $U_h(sl(2))$ is obvious. To calculate it explicitly, we introduce [5] derivations e_a dual to the 1-forms θ^a, defined by

$$e_1 x = y \qquad e_1 y = 0$$
$$e_2 x = 0 \qquad e_2 y = -y.$$

In terms of them the Laplace operator Δ_h can be written as

$$-\Delta_h \phi = e_1^2\phi + e_2^2\phi + e_2\phi \qquad \phi \in \mathcal{A}_h. \tag{5.1}$$

First we recall the calculation of the propagator in the commutative case. In the commutative limit Δ_h tends to the ordinary Laplace operator on the Lobachevsky plane:

$$\lim_{h\to 0} \Delta_h = \tilde{\Delta} = -\tilde{y}^2(\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2). \tag{5.2}$$

Here $(\tilde{x}, \tilde{y})$ are the commutative limits of the operators (x, y). The spectrum of Δ_h in the commutative limit is given [17] by the eigenvalue equation

$$\tilde{\Delta}\phi(\tilde{x}, \tilde{y}) = \lambda_{k,\kappa}\phi(\tilde{x}, \tilde{y}). \tag{5.3}$$

By the separation of variables $\phi(\tilde{x}, \tilde{y}) = f(\tilde{x})g(\tilde{y})$ one finds the differential equations

$$\partial_{\tilde{x}}^2 f(\tilde{x}) = -k^2 f(\tilde{x}) \tag{5.4}$$

$$\tilde{y}^2\partial_{\tilde{y}}^2 g(\tilde{y}) = (k^2\tilde{y}^2 - \lambda_{k,\kappa})g(\tilde{y}) \tag{5.5}$$

where $k \in \mathbb{R}$. We define κ^2 by

$$\lambda_{k,\kappa} = \kappa^2 + \tfrac{1}{4}.$$

The eigenvalues $\lambda_{k,\kappa}$ do not, in fact, depend on k and are infinitely degenerate. If we then set $z = \mathrm{i}k\tilde{y}$ and $g(\tilde{y}) = \sqrt{z}\, J(z)$, equation (5.5) becomes the Bessel equation

$$J''(z) + \frac{1}{z}J'(z) + \left(1 + \frac{\kappa^2}{z^2}\right)J(z) = 0. \tag{5.6}$$

A normalized set of eigenfunctions for the Laplace operator is given by

$$\phi_{k,\kappa}(\tilde{x}, \tilde{y}) = \mathrm{e}^{\mathrm{i}k\tilde{x}}\pi^{-3/2}\sqrt{\kappa \sinh \pi\kappa}\sqrt{\tilde{y}}K_{\mathrm{i}\kappa}(|k|\tilde{y}) \tag{5.7}$$

with $\kappa > 0$ and $k \neq 0$. The case $\kappa < 0$ can be excluded since

$$K_{-\nu}(|k|\tilde{y}) = K_\nu(|k|\tilde{y}).$$

The case $k = 0$ is also excluded since when $\tilde{y} \to 0$

$$K_{\mathrm{i}\kappa}(|k|\tilde{y}) \to \frac{1}{2}\Gamma(\mathrm{i}\kappa)\left(\frac{2}{|k|\tilde{y}}\right)^{\mathrm{i}\kappa} + \frac{1}{2}\Gamma(-\mathrm{i}\kappa)\left(\frac{2}{|k|\tilde{y}}\right)^{-\mathrm{i}\kappa}. \tag{5.8}$$

If we set $\tilde{x}^i = (\tilde{x}, \tilde{y})$ the completeness relation can be written as

$$\delta^{(2)}(\tilde{x}^i - \tilde{x}^{i\prime}) = \int_{-\infty}^{+\infty}\int_0^{\infty} \phi_{k,\kappa}(\tilde{x}, \tilde{y})\, \phi^*_{k,\kappa}(\tilde{x}', \tilde{y}')\, \mathrm{d}k\, \mathrm{d}\kappa \tag{5.9}$$

and the propagator of $(\Delta + \mu^2)$ is given by

$$G(\tilde{x}^i, \tilde{x}^{i\prime}) = \int_{-\infty}^{+\infty}\int_0^{\infty} \frac{\phi_{k,\kappa}(\tilde{x}, \tilde{y})\, \phi^*_{k,\kappa}(\tilde{x}', \tilde{y}')}{\kappa^2 + \frac{1}{4} + \mu^2}\, \mathrm{d}k\, \mathrm{d}\kappa. \tag{5.10}$$

Now consider the non-commutative case. Notice that although the classical Lobachevsky plane is invariant under the reflection $\tilde{x} \to -\tilde{x}$, this is no longer the case when $h \neq 0$. By again ordering any monomial in $\mathcal{A}$ in the form $\phi(x|y)$, one can formally separate the variables in the eigenvalue problem as before [4] and the eigenvalue equation can be decomposed into two differential equations. The equations for the factor $f(x)$ are given by

$$\begin{aligned} e_1^2 f(x) &= -L_+^2 y^2 f(x) \\ e_1^2 f(x) &= -L_-^2 f(x) y^2 \end{aligned} \tag{5.11}$$

where $L_\pm \in \mathbb{R}$. Since the commutation relations $[y, e_2]$ and $[\tilde{y}, \tilde{y}\partial_{\tilde{y}}]$ are of the same form, the differential equation for $g(y)$ has the same form as that of (5.5) even though the algebra has changed:

$$(e_2^2 + e_2)g(y) = (L_\pm^2 y^2 - \lambda_{k,\kappa})g(y).$$

Consider the functions

$$L_\pm(k) = \frac{e^{\pm 2hk} - 1}{2h}.$$

For any $k \in \mathbb{R}$ let e^{ikx} be defined as a formal power series in the element x. Then from the action of e_1 on x it follows that

$$e_1 e^{ikx} = iL_+(k)\, y e^{ikx} = iL_-(k)\, e^{ikx} y \tag{5.12}$$

where we have used

$$e^{ikx} f(y) = f(e^{2hk} y)\, e^{ikx}. \tag{5.13}$$

The solution of equation (5.11) is therefore given by

$$f(x) = e^{ikx} \qquad L_\pm = L_\pm(k). \tag{5.14}$$

A family of formal solutions of the eigenvalue equation on the quantum Lobachevsky plane which tends to normalized functions in the commutative limit is given for $k \neq 0$, $\kappa > 0$ by

$$\phi_{k,\kappa}(x, y) = \pi^{-3/2}\sqrt{\kappa \sinh \pi\kappa}\, e^{ikx} \sqrt{y}\, K_{i\kappa}(|L_-(k)|y). \tag{5.15}$$

Thus $L_-(k)$ plays the role of the linear momentum associated with x. Although $|k|$ remains invariant under the map $k \to -k$ this is not the case for $|L_-(k)|$, a fact which is a manifestation of the breaking of parity by the commutation relations. Moreover, the range of the momentum $L_-(k)$ in the x-direction appears at this point to be limited to the region $(-1/2h, \infty)$. We will come back to this in a moment.

Define the one-particle Hilbert space $\mathcal{H}^{(0)}$ to be generated by the (improper) basis

$$\phi_{k,\kappa}(x|y) = \pi^{-3/2}\sqrt{\kappa \sinh \pi\kappa}\, e^{ikx} \sqrt{y}\, K_{i\kappa}(|L_-(k)|y)$$

for $k \in (-\infty, \infty)$ and $\kappa > 0$. The inner product on this space should be invariant under $U_h(sl(2))$, which means that the star structure (2.20) of $U_h(sl(2))$ is induced by the adjoint of operators on the Hilbert space, as classically. As explained in the previous section, one such inner product is given by

$$\langle f(x|y), g(x|y)\rangle_{(0)} = \int : f(x|y)^* g(x|y) : d\mu \tag{5.16}$$

where the latter is the classical integral after normal ordering, i.e. x should be commuted to the left of y before taking the integral. It is clear that the Laplacian is (formally) a symmetric operator, since δ is the (formal) adjoint of d.

Now we can calculate the norm of the eigenstates as

$$\begin{aligned}\langle\phi_{k,\kappa},\phi_{k',\kappa'}\rangle_{(0)} &= \int :\phi_{k,\kappa}(x,y)^*\,\phi_{k',\kappa'}(x,y):\mathrm{d}\mu\\ &= \pi^{-3}\int\sqrt{\kappa\sinh(\pi\kappa)}\,\sqrt{\kappa'\sinh(\pi\kappa')}\\ &\quad :\sqrt{y}K^*_{\mathrm{i}\kappa}(|L_-(k)|y)\,\mathrm{e}^{-\mathrm{i}(k-k')x}\sqrt{y}K_{\mathrm{i}\kappa'}(|L_-(k')|y):\mathrm{d}\mu\\ &= \delta(k-k')\,\pi^{-3/2}\int_0^\infty \mathrm{d}y\,\frac{1}{y}\sqrt{\kappa\sinh(\pi\kappa)}\sqrt{\kappa'\sinh(\pi\kappa')}\\ &\quad\times K^*_{\mathrm{i}\kappa}(|L_-(k)|y)\,K_{\mathrm{i}\kappa'}(|L_-(k')|y)\\ &= \delta(k-k')\,\delta(\kappa-\kappa')\end{aligned}\tag{5.17}$$

as classically. The Hilbert space $\mathcal{H}^{(0)}$ can now be defined as the closure of normalizable wavepackets build from this 'basis' of eigenfunctions, which obviously define an isometry with the usual, undeformed Hilbert space of square-integrable functions.

Using two braided copies of $\mathcal{A}$ as in the previous section, the propagator can be written as

$$\begin{aligned}G(x^i,x^{i\prime}) &= \int_{-\infty}^{+\infty}\int_0^\infty \frac{\phi_{k,\kappa}(x,y)\,\phi^*_{k,\kappa}(x',y')}{\lambda_{k,\kappa}+\mu^2}\,\mathrm{d}k\,\mathrm{d}\kappa\\ &= \pi^{-3}\int_{-\infty}^{+\infty}\int_0^\infty(\lambda_{k,\kappa}+\mu^2)^{-1}\kappa\sinh(\pi\kappa)\\ &\quad\times\mathrm{e}^{\mathrm{i}kx}\sqrt{y}\,K_{\mathrm{i}\kappa}(|L_-(k)|y)\sqrt{y'}K^*_{\mathrm{i}\kappa}(|L_-(k)|y')\,\mathrm{e}^{-\mathrm{i}kx'}\,\mathrm{d}k\,\mathrm{d}\kappa\end{aligned}\tag{5.18}$$

$$\begin{aligned}&= \pi^{-3}\int_{-\infty}^{+\infty}\int_0^\infty(\lambda_{k,\kappa}+\mu^2)^{-1}\kappa\sinh(\pi\kappa)\\ &\quad\times\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}kx'}\sqrt{y}\,K_{\mathrm{i}\kappa}(|L_+(k)|y)\sqrt{y'}K^*_{\mathrm{i}\kappa}(|L_+(k)|y')\,\mathrm{e}^{2hk}\,\mathrm{d}k\,\mathrm{d}\kappa.\end{aligned}\tag{5.19}$$

Here we have used (5.13), the identity $|L_-(k)|\,\mathrm{e}^{2hk}=|L_+(k)|$ and the fact that the commutation relations (3.1) between y and x' are the same as those between y' and x'.

As is shown in appendix A, the commutation relations (3.1) between x and x' imply the following identity:

$$\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}kx'}=\mathrm{e}^{\mathrm{i}L_+(k)\,\delta x}\tag{5.20}$$

where we recall $\delta x=x-x'$. Together with (5.13), it follows

$$\begin{aligned}G(x^i,x^{i\prime}) &= \pi^{-3}\int_{-\infty}^{+\infty}\int_0^\infty(\lambda_{k,\kappa}+\mu^2)^{-1}\kappa\sinh(\pi\kappa)\\ &\quad\times\mathrm{e}^{\mathrm{i}L_+(k)\,\delta x}\sqrt{y}K_{\mathrm{i}\kappa}(|L_+(k)|y)\sqrt{y'}K^*_{\mathrm{i}\kappa}(|L_+(k)|y')\,\mathrm{e}^{2hk}\,\mathrm{d}k\,\mathrm{d}\kappa.\end{aligned}\tag{5.21}$$

Now recall that the subalgebra generated by $\overline{y}$, δx, δy is Abelian. Thus we can treat it as an ordinary function algebra, and change variables to $p=L_+(k)$, $\mathrm{d}p=\mathrm{e}^{2hk}\,\mathrm{d}k$. We have then

$$\begin{aligned}G(x^i,x^{i\prime}) &= \pi^{-3}\int_{-1/2h}^{+\infty}\mathrm{d}p\int_0^\infty\mathrm{d}\kappa\,(\lambda_{p,\kappa}+\mu^2)^{-1}\kappa\sinh(\pi\kappa)\\ &\quad\times\mathrm{e}^{\mathrm{i}p\delta x}\sqrt{y}K_{\mathrm{i}\kappa}(|p|y)\sqrt{y'}K^*_{\mathrm{i}\kappa}(|p|y').\end{aligned}\tag{5.22}$$

Recall that $\lambda_{p,\kappa}$ does not depend on p. The integrand is therefore exactly the same as classically (see (5.10) and (5.7)); only the integration limit of p has changed.

This is actually a rather strange result. Since the Laplacian is invariant under $U_h(sl(2))$, one should expect that the propagator would also invariant under this algebra (this is made more explicit in appendix B), which implies that it is a function of the invariant distance, which is the same as classically as we have seen. We just found that it is 'almost', but not quite: if the integration limits were the same as classically, it would of course be invariant, but the integration bound $-1/2h$ for p spoils invariance. How is this possible?

The only explanation seems to be that the representation of $U_h(sl(2))$ on the Hilbert space generated by the eigenfunctions (5.15) is not a $*$-representation, i.e. the generators of $U_h(sl(2))$ are not represented as (anti)self-adjoint operators. A similar phenomenon is known to happen in the case of the q-deformed quantum line [8], where one has to consider reducible Hilbert space representations in order to obtain self-adjoint representations of the quantum algebra.

In fact, we can find such an 'extended' Hilbert space here as well. Let

$$\phi^{(n)}_{k,\kappa}(x|y) = \pi^{-3/2}\sqrt{\kappa\sinh\pi\kappa}\,\mathrm{e}^{\mathrm{i}kx}\sqrt{y}\,K_{\mathrm{i}\kappa}(|L_-(k)|y)$$
$$\text{for}\qquad k\in(-\infty,\infty)+\frac{\mathrm{i}n\pi}{2h}\qquad\text{and}\qquad \kappa>0 \tag{5.23}$$

and let $\mathcal{H}^{(n)}$ be the closure of normalizable wavepackets built from (5.23), with an inner product defined as

$$\langle f(x|y), g(x|y)\rangle_{(n)} := \int : f(x,y)^*\,\mathrm{e}^{n\pi x/h}g(x,y) : \mathrm{d}\mu. \tag{5.24}$$

It is easy to see that this inner product is invariant under the sub-Hopf algebra of $U_h(sl(2))$ generated by $\{G^2, J^3G, J^-G\}$ which commutes with $\mathrm{e}^{n\pi x/h}$, and that the Laplacian is still symmetric since $\mathrm{d}\mathrm{e}^{n\pi x/h}=0$.

As the Hilbert space, $\mathcal{H}^{(n)}$ is of course equivalent to $\mathcal{H}^{(0)}$, but not as a representation of $U_h(sl(2))$. For example, consider the above 'plane-wave' states in $\mathcal{H}^{(1)}$: they are the eigenstates of the Laplacian with momentum $L_-(k)$ in the x-direction in the interval $(-\infty,-1/2h)$, which were 'missing' above. We can calculate the inner product on $\mathcal{H}^{(1)}$:

$$\begin{aligned}
\left\langle\phi^{(1)}_{k,\kappa},\phi^{(1)}_{k',\kappa'}\right\rangle_{(1)} &= \int : \phi^{(1)}_{k,\kappa}(x,y)^*\,\mathrm{e}^{n\pi x/h}\phi^{(1)}_{k',\kappa'}(x,y) : \mathrm{d}\mu\\
&= \pi^{-3}\int \sqrt{\kappa\sinh(\pi\kappa)}\sqrt{\kappa'\sinh(\pi\kappa')}\\
&\qquad : \sqrt{y}K^*_{\mathrm{i}\kappa}(|L_-(k)|y)\,\mathrm{e}^{-\mathrm{i}(k-k')x}\sqrt{y}K_{\mathrm{i}\kappa'}(|L_-(k')|y) : \mathrm{d}\mu\\
&= \delta(k-k')\,\pi^{-3/2}\int_0^\infty \mathrm{d}y\,\frac{1}{y}\sqrt{\kappa\sinh(\pi\kappa)}\,\sqrt{\kappa'\sinh(\pi\kappa')}\\
&\qquad\times K^*_{\mathrm{i}\kappa}(|L_-(k)|y)\,K_{\mathrm{i}\kappa'}(|L_-(k')|y)\\
&= \delta(k-k')\,\delta(\kappa-\kappa').
\end{aligned}\tag{5.25}$$

One can now repeat the calculation (5.19) for $\mathcal{H}^{(1)}$,

$$\begin{aligned}
G^{(1)}(x^i,x^{i\prime}) &= \int\mathrm{d}k\int_0^\infty\mathrm{d}\kappa\,\frac{\phi^{(1)}_{k,\kappa}(x,y)\,\phi^{(1)*}_{k,\kappa}(x',y')\,\mathrm{e}^{n\pi x/h}}{\lambda_{k,\kappa}+\mu^2}\\
&= \pi^{-3}\int\mathrm{d}k\int_0^\infty\mathrm{d}\kappa\,\left(\lambda_{k,\kappa}+\mu^2\right)^{-1}\kappa\sinh(\pi\kappa)\,\cdot\\
&\qquad\times\mathrm{e}^{\mathrm{i}L_+(k)\,\delta x}\sqrt{y}K_{\mathrm{i}\kappa}(|L_+(k)|y)\sqrt{y'}K^*_{\mathrm{i}\kappa}(|L_+(k)|y')\,\mathrm{e}^{2hk}.
\end{aligned}\tag{5.26}$$

Changing variables again to $p = L_+(k)$, $\mathrm{d}p = \mathrm{e}^{2hk}\,\mathrm{d}k$, we obtain

$$G^{(1)}(x^i, x^{i\prime}) = \pi^{-3} \int_{-\infty}^{-1/2h} \mathrm{d}p \int_0^{\infty} \mathrm{d}\kappa \left(\lambda_{p,\kappa} + \mu^2\right)^{-1} \kappa \sinh(\pi\kappa)\, \mathrm{e}^{\mathrm{i}p\delta x}$$
$$\times \sqrt{y} K_{\mathrm{i}\kappa}(|p|y) \sqrt{y'} K^*_{\mathrm{i}\kappa}(|p|y'). \tag{5.27}$$

This is precisely the missing piece in order to obtain an invariant propagator. We therefore define the 'extended' Hilbert space to be the direct orthogonal sum

$$\mathcal{H} := \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}. \tag{5.28}$$

Then on $\mathcal{H}$, the propagator is invariant and exactly as classically,

$$G^{\mathcal{H}}(x^i, x^{i\prime}) = \pi^{-3} \int_{-\infty}^{\infty} \mathrm{d}p \int_0^{\infty} \mathrm{d}\kappa\, (\lambda_{p,\kappa} + \mu^2)^{-1} \kappa \sinh(\pi\kappa)\, \mathrm{e}^{\mathrm{i}p\delta x}$$
$$\times \sqrt{y} K_{\mathrm{i}\kappa}(|p|y) \sqrt{y'} K^*_{\mathrm{i}\kappa}(|p|y'). \tag{5.29}$$

6. Discussion

It was found in [12] that the propagator on the h-deformed Lobachevsky plane is finite if one uses the usual, 'unbraided' tensor product; similarly for a non-commutative flat plane. However, this tensor product 'breaks' the invariance under the quantum group.

In this paper, we have first seen that the h-deformation is not a trivial deformation, even though it is just a twist of the undeformed case; it turns out that the structure of the Hilbert space is modified. If this is done properly and the covariant, braided tensor product is used, then the propagator turns out to be the same as classically; in particular, it is divergent. This is certainly disappointing, since the main reason for considering the deformed manifolds is to 'smear' the points, thereby regularizing the UV divergences. It is not entirely clear how to understand this result. It could be that the algebra is simply not non-commutative enough. Another interpretation might be that the identification of the 'distances' δx, δy in the braided tensor product is not satisfactory, in particular, since they do not commute with the 'average' values $\overline{x}$, $\overline{y}$. See also the related discussion in [12]. In other words, the non-trivial (braided) commutation relations between different copies of the quantum space which are required by the covariance under a quantum group imply some kind of interaction. It seems that the physical meaning of the braided tensor product is not completely understood, and deserves further investigation.

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Appendix A

We prove that $\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}kx'} = \mathrm{e}^{\mathrm{i}L_+(k)\,\delta x}$, where $[x, x'] = 2\mathrm{i}h(x - x')$.

From $[x, \delta x] = -2\mathrm{i}h\delta x$ it follows that $[x, \ldots [x, x'] \ldots] = -(-2\mathrm{i}h)^n \delta x$ for n commutators, and thus

$$\mathrm{e}^{\mathrm{i}kx} x' \mathrm{e}^{-\mathrm{i}kx} = x' - (\mathrm{e}^{2hk} - 1)\, \delta x. \tag{A.1}$$

Let $f_k(x, x') = e^{ikx} e^{-ikx'}$. Using (A.1), we find

$$\frac{d}{dk} f_k(x, x') = ix e^{ikx} e^{-ikx'} - e^{ikx} ix' e^{-ikx'}$$
$$= i(x - x' + (e^{2hk} - 1)\,\delta x)\, f_k(x, x')$$
$$= e^{2hk} i\delta x f_k(x, x'). \tag{A.2}$$

Consider $dp(k)/dk = e^{2hk}$, with the solution $p(k) = \frac{1}{2h}(e^{2hk} - 1)$. Then (A.2) is equivalent to

$$\frac{d f_p(x, x')}{dp} = i\delta x f_p(x, x')$$

with the solution $f_p(x, x') = e^{ip\delta x}$. Therefore,

$$e^{ikx} e^{-ikx'} = e^{(i/2h)(e^{2hk}-1)\,\delta x} = e^{iL_+(k)\,\delta x}. \tag{A.3}$$

Appendix B

We explain without mathematical rigour why the propagator should be invariant under $U_h(sl(2))$ if the latter is implemented via a $*$-representation. We assume the spectral decomposition $\mathbf{1} = \int_n \phi_n \otimes \phi_n^*$ where $\Delta\phi_n = \lambda_n \phi_n$.

For $u \in U_h(sl(2))$, one has

$$u \cdot G(z, z') = u\left(\int_n \lambda_n^{-1} \phi_n(z)\, \phi_n^*(z')\right)$$
$$= \int_n \lambda_n^{-1} (u_{(1)} \cdot \phi_n(z))(u_{(2)} \cdot \phi_n^*(z')) \tag{B.1}$$

where z stands for (x, y). Let

$$u \cdot \phi_n = \int_l \phi_l \pi_{ln}(u).$$

Since π is a $*$-representation one has $\pi_{ln}(u^*) = \pi_{nl}^*(u)$. We claim that this implies that

$$u \cdot \phi_k^* = \int_l \pi_{kl}(Su)\, \phi_l^* \tag{B.2}$$

where Su is the antipode of u. Indeed, using $U_h(sl(2)) \ltimes \mathcal{A}$, one has

$$u \cdot \phi_k^* = u_{(1)} \phi_k^* S u_{(2)} = \left(S^{-1}(u_{(2)}^*)\, \phi_k u_{(1)}^*\right)^* = \left((S^{-1}u^*) \cdot \phi_k\right)^*$$
$$= \left(\int_l \phi_l \pi_{lk}(S^{-1}u^*)\right)^* = \int_l \phi_l^* \pi_{lk}^*(S^{-1}u^*) = \int_l \pi_{kl}(Su)\, \phi_l^* \tag{B.3}$$

where the $*$-representation property was used in the last line, as well as standard identities for $*$-Hopf algebras. Now we can conclude that

$$u \cdot G(z, z') = \int_{n,l,k} \lambda_n^{-1} \phi_l(z)\, \pi_{ln}(u_{(1)})\, \pi_{nk}(Su_{(2)})\, \phi_k^*(z')$$
$$= \epsilon(u) \int_{n,l,k} \phi_l(z)\, \delta_{lk} \phi_k^*(z') = \epsilon(u)\, G(z, z'). \tag{B.4}$$

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